Closing Wed: HW_8 (8.3)
10 center of mass questions.
(There is no 8A, 8B, 8C, just HW_8).
Midterm 2 will be returned Tuesday.

Chapter 8: More Applications.

8.1 Arc Length (length along a curve)

8.3 Center of Mass

For your own interest you can read:
8.2: Surface area
8.3: hydrostatic (water) pressure & force
8.4: economics and biology apps
8.5: probability apps (bell curve)

8.3 Center of Mass

Goal: Given a thin plate (a *lamina*) where the mass is uniformly distributed, we find the center of mass (*centroid*).

Motivation:

If you are given ***n* points**

$(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ with masses

$m_1, m_2, \ldots, m_n$

then

$$M = \text{total mass} = \sum_{i=1}^{n} m_i$$

$$M_y = \text{moment } \textit{about } y \text{ axis} = \sum_{i=1}^{n} m_i x_i$$

$$M_x = \text{moment } \textit{about } x \text{ axis} = \sum_{i=1}^{n} m_i y_i$$

$$\bar{x} = \frac{m_1 x_1 + \cdots + m_n x_n}{m_1 + \cdots + m_n} = \frac{M_y}{M}$$

$$\bar{y} = \frac{m_1 y_1 + \cdots + m_n y_n}{m_1 + \cdots + m_n} = \frac{M_x}{M}$$

Derivation: (don't need to write this)

Consider a thin plate with uniform density

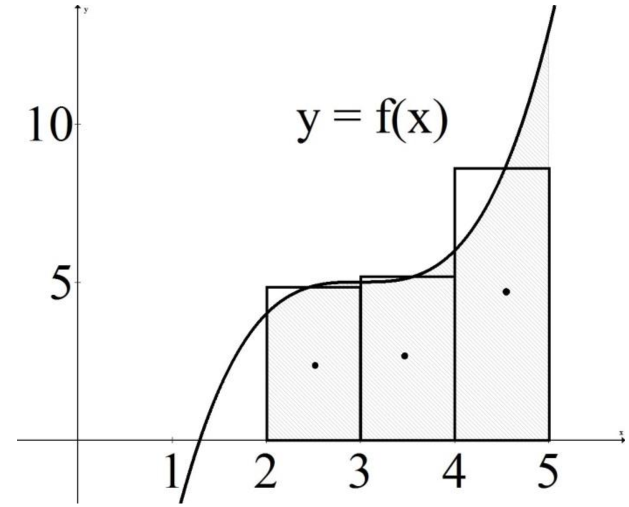
ρ = mass/area = a constant

1. Break into *n* subdivision
   $$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$
2. Draw a (midpoint) rectangle for each
3. The center of mass of each rectangle
   $(\bar{x}_i, \bar{y}_i)$, **Note**: $\bar{y}_i = \frac{1}{2} f(\bar{x}_i)$.
4. Mass of each rectangle:
   $m_i = p(Area) = pf(x_i)\Delta x$.
5. Now use the formula for *n* points.
6. Take the limit.

$$\bar{x} = \frac{\lim_{n\to\infty} \sum_{i=1}^{n} pf(x_i)\Delta x\, x_i}{\lim_{n\to\infty} \sum_{i=1}^{n} pf(x_i)\Delta x} = \frac{p\int_a^b xf(x)dx}{p\int_a^b f(x)dx}$$

$$\bar{y} = \frac{\lim_{n\to\infty} \sum_{i=1}^{n} pf(x_i)\Delta x\, \frac{1}{2} f(x_i).}{\lim_{n\to\infty} \sum_{i=1}^{n} pf(x_i)\Delta x} = \frac{p\int_a^b \frac{1}{2}(f(x))^2 dx}{p\int_a^b f(x)dx}$$

Visual Example of Derivation



Above: f(x) = (x-3)^3 *+ 5* from x = 2 to x = 5.

Assume ρ = 7 kg/m^2 = density

If n = 3, then you get: $\Delta x = \frac{5-2}{3} = 1,$

$$\bar{x}_1 = 2.5, \bar{y}_1 = \frac{1}{2} f(2.5)$$
$$m_1 = 7f(2.5)\Delta x$$

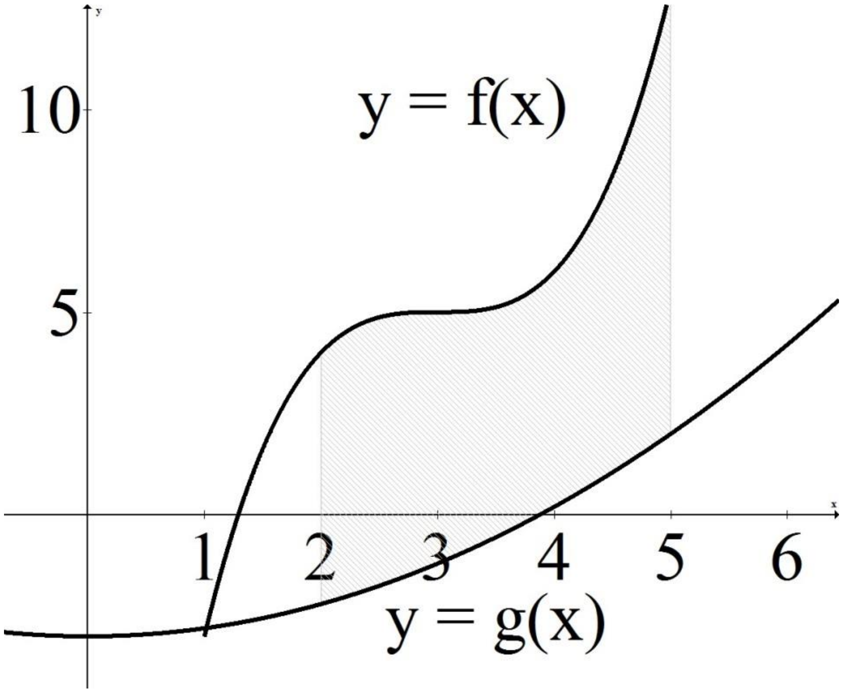
$$\bar{x} = \frac{7\int_2^5 xf(x)dx}{7\int_2^5 f(x)dx} = \frac{\int_2^5 x((x-3)^3+5)\,dx}{\int_2^5 (x-3)^3 + 5\,dx}$$

$$\bar{y} = \frac{7\int_2^5 \frac{1}{2}(f(x))^2 dx}{7\int_2^5 f(x)dx} = \frac{\int_2^5 \frac{1}{2}((x-3)^3+5)^2\,dx}{\int_2^5 (x-3)^3 + 5\,dx}$$

Example:

Find the center of mass (centroid) of a thin plate with uniform density $\rho = 2\ kg/m^2$ that looks like the region bounded by $y = 4 - x^2$ and the x-axis.

If the region is bounded between two curves,



what changes in derivation

$$\bar{x} = \frac{p \int_a^b x(f(x) - g(x))dx}{p \int_a^b f(x) - g(x)\, dx}$$

$$\bar{y} = \frac{p \int_a^b \frac{1}{2}[(f(x))^2 - (g(x))^2]dx}{p \int_a^b f(x) - g(x)dx}$$

Example:

Find the center of mass (centroid) of a thin plate with uniform density ρ = 3 kg/m^2 that looks like the region bounded by $y = x$ and $y = \sqrt{x}$.

Just for your own interest:

Theorem of Pappus

The volume of a solid of revolution is equal to the product of the area of the region, A, and the distance traveled by the center of mass of the region around the axis of rotation, d. (Note: $d = 2\pi\bar{x}$)

Thus, Volume = (Area)$2\pi\bar{x}$

Proof

Using the shell method, we get:

$$\text{Volume} = \int_a^b 2\,\pi x\big(f(x) - g(x)\big)dx$$

$$= 2\pi \int_a^b x\,\big(f(x) - g(x)\big)dx$$

From today:

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x))dx}{\text{Area}}, \qquad \text{so}$$

$$\int_a^b x(f(x) - g(x))dx = \bar{x}(Area).$$

Example:

Find the volume of the torus.